Grade 7/8 Math Circles

October 17/18/19/23, 2023

Introduction to Proofs - Problem Set

1. State the hypothesis and conclusion of the following statements.

   (a) If x is positive, then $x - 7$ is positive.

   (b) If I walk to school today, then Jacob takes the bus to school today.

2. Determine why this proof does not hold true.

   Hypotheses:

   - John is responsible for walking his dog at least twice per week.
   - John didn't walk his dog on Monday, Tuesday, or Wednesday.

   Conclusion: Therefore John walked his dog on Friday.

3. Disprove the following statements.

   (a) If x is positive, then $x - 10$ is negative.

   (b) If x is positive, then $x - 10$ is positive.

   (c) If x and y are even, and z is odd, then $x + y + 2z$ is odd.

4. Fully factor the following expressions.

   (a) $15x + 9xy$

   (b) $10a + 2b - 8$

   (c) $x^2 + x$

5. Expand the following expressions.

   (a) $3(10x + 9y - 4)$

   (b) $5a(2a + 3b)$

   (c) $2xy(5x + 1)$

6. Prove the following statements.

   (a) If x is even and y is odd, then xy is even.

   (b) If x is even and y is odd, then $x + y$ is odd.

   (c) If x, y, and z are all odd, then $x + y + z$ is odd.

7. Prove the following statement. Is the converse true? If it is true, prove it. If it is false, give a counter-example.

   If both x and y are even, then xy is even.

8. CHALLENGE PROBLEM 1

   Prove the divisibility by 3 rule for three-digit numbers. That is, prove the following statement.

   A three-digit number is divisible by 3 if and only if the sum of its digits is divisible by 3.

   HINT: The proof from Example 10 in the lesson is a good guide for this question.

CHALLENGE PROBLEM 2 is on the next page.

9. CHALLENGE PROBLEM 2

This final question is a proof of a very famous theorem, but you will need to learn more about the distributive property to complete this challenge problem. Below is a mini-lesson to help with this.

FOIL mini-lesson

We need to learn to expand an expression of the form $(a+b)(c+d)$ and will do so with the help of the acronym FOIL. This is a step up from before where we expanded $a(b+c)$. FOIL stands for First, Outside, Inside, Last.

- First: Multiply the first term in each set of brackets. So, ac.
- Outside: Multiply the outside terms of the brackets. So, ad.
- Inside: Multiply the inside terms of the brackets. So, bc.
- Last: Multiply the last term in each set of brackets. So, bd.
- That add them all together. So, $(a+b)(c+d) = ac + ad + bc + bd$.

Here are some concrete examples.

(a)

$$\begin{aligned}(3+2)(1+5) &= (5)(6) && \text{(using BEDAMS)}\\ &= 30\\ (3+2)(1+5) &= (3)(1)+(3)(5)+(2)(1)+(2)(5) && \text{(using FOIL)}\\ &= 3+15+2+10\\ &= 30\end{aligned}$$

(b)

$$\begin{aligned}(a+2)(a+b) &= (a)(a)+(a)(b)+(2)(a)+(2)(b) && \text{(using FOIL)}\\ &= a^2+ab+2a+2b\end{aligned}$$

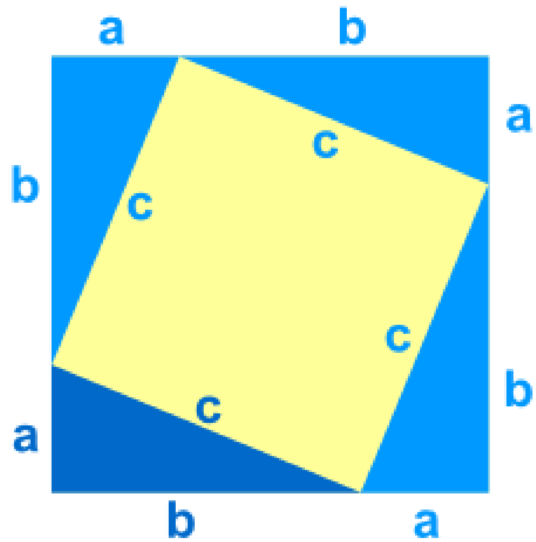

Figure 1: Image retrieved from Math is Fun

Now you are ready for the proof! We need to think about the area of the large square in two ways.

(a) The side length of the large square is $a + b$. The area of a square is the side length mutliplied by itself. So, area $= (a + b)(a + b)$.

Expand $(a + b)(a + b)$.

(b) The area of the large square can also be thought of as the area of the yellow square plus the area of the four triangles.

Recall that the area of a triangle is $\frac{(\text{base}) \times (\text{height})}{2}$.

Write the area of the large square as the area of the yellow square plus the area of the four triangles.

(c) Your answer to part (a) and part (b) are measure the same area, so they are equal! Set them equal to each other.

Then try to show that $a^2 + b^2 = c^2$

Congratulations! You just *proved* the Pythagorean Theorem!

Here's a slider made in GeoGebra that demonstrates the Pythagorean Theorem! `https://www.geogebra.org/m/afygf9dq`.